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GAME PROBLEM OF THE HARD CONTACT OF TWO POINTS WITH AN IMPULSE THRUST IN A LINEAR CENTRAL FIELD

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We consider a differential game [1-3] directly related to [4], where an analogous problem was analyzed for points under the action of controls alone and to [5], where the problem was investigated of the "soft" contact (with respect to coordinates and velocities) of points in a linear central field. In the present paper we solve the problem of the minimax time up to the "hard" contact (with respect to coordinates) of two points (players) with masses m_1 and m_2, moving under the action of position forces $F_1 = -\omega^2 m_1 r_1$ and $F_2 = -\omega^2 m_2 r_2$ (r_1, r_2 are radius vectors of the points relative to the center of attraction) and of controls $f_1 = m_1 u$ and $f_2 = -m_2 v$ arbitrary in direction and bounded with respect to the total momentum. The first player minimizes, while the second maximizes, the time up to the hard contact. The whole space of possible positions is separated into two regions. In the first region we find the optimal controls of both players and the minimax time up to the "hard" contact. In the second region we form the second player's control which he uses avoiding contact under any action of the first player.

1. The equations of relative motion ($x = r_1 - r_2$, $y = r_1^{\cdot} - r_2^{\cdot}$), after a scale change in length and in time reducing to the equality $\omega = 1$, have the form

$$x^{\cdot} = y, \quad y^{\cdot} = -x + u + v, \quad \mu^{\cdot} = -|u|, \quad \nu^{\cdot} = -|v| \tag{1.1}$$

$$\mu \geqslant 0, \qquad \nu \geqslant 0 \tag{1.2}$$

where $x,\ y,\ u,\ v$ are three-dimensional vectors, $|u|,\ |v|$ are Euclidean moduli, and $\mu \geqslant 0,\ \nu \geqslant 0$ are numbers. The constraints $\mu \geqslant 0,\ \nu \geqslant 0$ and the equations $\mu^{\cdot} = -|u|,\ \nu^{\cdot} = -|v|$ are equivalent to "impulse" constraints on the controls $u,\ v$ of the first and second players. These constraints have the form

$$\mu^{\circ} - \int_0^{\tau} |u|\,dt = \mu^{(1)}(\tau) \geqslant 0 \qquad (1.3)$$

$$\nu^{\circ} - \int_0^{\tau} |v|\,dt = \nu^{(2)}(\tau) \geqslant 0$$

and permit instantaneous changes in vector y and in the numbers $\mu,\ \nu$ by the formulas

$$y^{(2,1)} = y + \mu_1 + \nu_2, \qquad \mu^{(2,1)} = \mu - |\mu_1|, \qquad \nu^{(2,1)} = \nu - |\nu_2| \qquad (1.4)$$

where $\mu_1,\ \nu_2$ are three-dimensional vectors. In this case we will consider the impulse controls $u = \mu_1\delta,\ v = \nu_2\delta$.

We call the vector $w = [x,\ y,\ \mu,\ \nu]$, defined by the collection of arguments indicated, the position and we introduce into consideration the vectors

$$w^{(1)} = [x, y^{(1)} = y + \mu_1(w),\ \ \mu^{(1)} = \mu - |\mu_1(w)|;\ \ \nu^{(1)} = \nu]$$
$$w^{(2)} = [x, y^{(2)} = y + \nu_2(w),\ \ \mu^{(2)} = \mu,\ \ \nu^{(2)} = \nu - |\nu_2(w)|\,]$$
$$w^{(2,1)} = [x, y^{(2,1)} = y^{(2)} + \mu_1(w^{(2)}),\ \ \mu^{(2,1)} = \mu - |\mu_1(w^{(2)})|;\ \ \nu^{(2,1)} = \nu^{(2)}]$$
$$w^{(1,2)} = [x, y^{(1,2)} = y^{(1)} + \nu_2(w^{(1)}),\ \ \mu^{(1,2)} = \mu^{(1)},\ \ \nu^{(1,2)} = \nu - |\nu_2(w^{(1)})|\,]$$

where $w^{(1)},\ w^{(2)}$ denote the results of impulse actions of the first and second players, which can be realized as functions of vector w. The vector $w^{(2,1)}$ ($w^{(1,2)}$) represents the result of the impulse actions at first of the second (first) and next of the first (second) players. Let the vector $w^{(2,1)}$ $(t \geqslant 0)$, $(w^{(1,2)}$ $(t \geqslant 0))$ be given as a function of time. The initial value of w is called the position at $t = 0$, while the left limit $w^{(2,1)}(\tau - 0)$, $(w^{(1,2)}(\tau - 0))$ is called the position at $t = \tau > 0$. The pair $u\ (w^{(2)}, v),\ v\ (w)$ and the trajectory $w^{(2,1)}$ $(t \geqslant 0;\ \{u\ (w^{(2)},\ v),\ v\ (w)\};\ w\ (0))$ corresponding to it are said to be admissible if for all t the trajectory is unique, right continuous, and satisfies constraints (1.2), and if Eq. (1.1) is satisfied for almost all t. Furthermore, on each finite interval $0 \leqslant t \leqslant t_1$ the trajectory can admit of a finite number of jumps in accord with formulas (1.4). The admissible pair $u\ (w),\ v\ (w^{(1)},\ u)$ and the trajectory $w^{(1,2)}$ $(t \geqslant 0,\ \{u\ (w), v\,(w^{(1)},\ u)\}\ \ w\ (0))$ are defined analogously. We call the set $M\ [|x| = 0]$ the game termination set (the set of hard contact) and we examine two problems on admissible pairs.

Problem 1. Find the pair $u^{\circ}\ (w^{(2)},\ v),\ v^{\circ}\ (w)$ such that the time $T\ [u,\ v]$ of the first hitting of the position onto M would satisfy the estimates

$$T\ [u^{\circ}(w^{(2)},\ v),\ v\,(w)] \leqslant T\ [u^{\circ}(w^{(2)},\ v),\ v^{\circ}(w)] \leqslant T\ [u\,(v^{(2)},\ v),\ v^{\circ}(w)]$$

Problem 2. Find the control $v_0\ (w^{(1)},\ u)$ such that the trajectory corresponding to any pair $u\ (w),\ v_0\ (w^{(1)},\ u)$ would not hit upon set M in finite time.

Let $y_\alpha,\ y_\beta$ be the projections of vector y onto vector x and onto a plane perpendicular to x. We introduce the right unit triple of unit vectors $j_\alpha,\ j_\beta,\ j_\gamma$ by the formulas

$$j_\alpha = x/|x|, \quad j_\beta = y_\beta/|y_\beta|, \quad w \in D_2\ [|x| > 0,\quad |y_\beta| > 0]$$
$$j_\alpha = x/|x|;\quad j_\beta,\ j_\gamma\ \text{arbitrary};\qquad w \in D_2\ [|x| > 0;\ |y_\beta| = 0]$$

Denoting the projections of a vector onto the unit vectors by the subscripts α, β, γ we obtain in regions D_1, D_2 the corollaries of Eq. (1.1)

$$|x|^{\cdot} = y_\alpha, \quad y_\alpha^{\cdot} = -|x| + u_\alpha + v_\alpha + y_\beta^2|/|x|, \qquad (1.5)$$

$$|y_\beta|^{\cdot} = u_\beta + v_\beta - y_\alpha|y_\beta/|x|$$

$$|x|^{\cdot} = y_\alpha, \quad y_\alpha^{\cdot} = -|x| + u_\alpha + v_\alpha, \quad |y_\beta|^{\cdot} = \sqrt{(u_\beta + v_\beta)^2 + (u_\gamma + v_\gamma)^2}$$

and the corollaries of Eqs. (1.4)

$$y_\alpha^{(2,1)} = y_\alpha + \mu_{1\alpha} + \nu_{2\alpha}$$

$$|y_\beta^{(2,1)}| = [(|y_\beta| + \mu_{1\beta} + \nu_{2\beta})^2 + (\mu_{1\gamma} + \nu_{2\gamma})^2]^{1/2}$$

Intuitively it is clear that the solution depends only on the quantities $|x|$, y_α, $|y_\beta|$, $\xi = \mu - \nu$. For the collection of them we retain the notation w.

2. The motion by virtue of system (1.1) with $u = v = 0$ admits of the first integrals of energy and of moment of momentum

$$2h(w) = y_\alpha^2 + y_\beta^2 + x^2, \qquad k(w) = |y_\beta||x|$$

while the minimal value $|y_\beta|^\circ$ of the quantity $|y_\beta|$ for the uncontrolled motion has the form

$$|y_\beta|^\circ = \sqrt{h(w) - \sqrt{h^2(w) - k^2(w)}}$$

The function $z(w) = \xi - |y_\beta(w)|^\circ$ divides the region $D = D_1 \cup D_2$ into the two regions

$$D^\circ[|x| > 0, \quad z(w) \geqslant 0], \qquad D_0[|x| > 0, \quad z(w) < 0]$$

Lemma 2.1. The pair $u(w^{(2)}, v)$, $v(w) = 0$ does not increase the function $z(w^{(2)})$, while the pair $u(w) = 0$, $v(w^{(1)}, u)$ does not decrease the function $z(w)$.

Proof. It is obvious that the equality $w^{(2)} = w$ is valid when $v(w) = 0$. Let us consider some finite control $u = u(w, v)$. The right derivative $z^{\cdot}(w, u, v = 0)$ has the form

$$z^{\cdot}(w, u, 0) = -|u| + \lambda_\alpha v_\alpha + \lambda_\beta u_\beta \quad \text{for } z(w) \neq 0$$

$$\lambda_\alpha = -\frac{y_\alpha z(w)}{2\sqrt{h - k^2}}, \qquad \lambda_\beta = -\frac{y_\beta^2 z(w) + k|x|/z(w)}{2\sqrt{h - k^2}}$$

$$h = h(w), \qquad k = k(w)$$

It is not difficult to see that the estimate $z^{\cdot} \leqslant 0$ follows from the estimate $\lambda_\alpha^2 + \lambda_\beta^2 - 1 \leqslant 0$. We compute the quantity

$$\lambda_\alpha^2 + \lambda_\beta^2 - 1 = -1 + \frac{z^2(2h - x^2) + (2h - x^2) - 2k^2}{4(h - k^2)}$$

where $z = z(w)$. The factor in the brackets increases monotonically with $|x|$ and reaches a maximum, equal to $4(h - k^2)$, for $|x| = k / z(w)$ which is equal to the maximal value of $|x|$ on the uncontrolled trajectory. Hence follow the estimates $\lambda_\alpha^2 + \lambda_\beta^2 - 1 \leqslant 0$ and $z^{\cdot} \leqslant 0$ for $z(w) \neq 0$. When $z(w) = 0$ the derivative $z^{\cdot}$ has the form

$$z^{\cdot} = -|u| + |x|\sqrt{u_\beta^2 + u_\gamma^2} / \sqrt{2h} \leqslant 0$$

In summary, we have established the estimate $z^{\cdot}(w, u, v = 0) \leqslant 0$ for any finite u. For impulse $u(w) = \mu_1\delta$ the estimate $\Delta z^{(1)}{}_1 = z(w^{(1)}) - z(w) \leqslant 0$ can be obtained by applying the theorem on the mean. The proof of the estimates

$$z^{\cdot}(w, u = 0, v) \geqslant 0, \qquad \Delta z^{(2)} = z(w^{(2)}) - z(w) \geqslant 0$$

is obtained analogously and completes the proof of Lemma 2.1.

Theorem 2.1. If $w(0) \in D_0$, the control

$$v_0(w^{(1)}, u) = 0, \qquad w^{(1)} \in D_{0,1} = D_0 \cap [\mu^{(1)} - \nu = \xi^{(1)} > 0]$$
$$v_0(w^{(1)}, u) = |u| j_\beta, \qquad w^{(1)} \in D_{0,2} = D_0 \cap [\xi^{(1)} = 0]$$
$$v_0(w^{(1)}, u) = -\xi^{(1)}\delta j_\beta, \qquad w^{(1)} \in D_{0,3} = D_0 \cap [\xi^{(1)} < 0]$$

solves Problem 2.

Proof. Let us assume to the contrary that for some $t = \tau > 0$ the equality $|x(\tau)| = 0$ is fulfilled on an admissible trajectory. This trajectory cannot lie, for $0 < t < \tau$, wholly in the region $D_{0,1}$ because the estimate $\xi(t) > 0$ and the estimate $z(w(t)) \leqslant z(w(0)) < 0$, being a consequence of the conditions $v_0 = 0$ for $w \in D_{0,1}$, and Lemma 2.1 lead to a contradiction. This means that for some t_1 from the interval $[0, \tau]$ the inclusion $w^{(1)}(t_1) \in D_{0,2}$ or the inclusion $w^{(1)}(t_1) \in D_{0,3}$ should be valid. In the first case it is necessary to satisfy the estimate $|y_\beta^{(1)}(t_1)| > 0$, while in the second, the estimate $|y_\beta^{(1,2)}(t_1)| > 0$. Further, the control $v_0(w^{(1)}, u)$ leads the trajectory $w^{(1,2)}(t \geqslant t_1)$ along region $D_{0,2}$ while preserving the estimate $|y_\beta^{(1,2)}(t)| \; |x^{(1,2)}(t)| \geqslant |y_\beta^{(1,2)}(t_1)| \; |x^{(1,2)}(t_1)| > 0$, and hitting onto M is impossible. The proof is completed.

3. In the region $D^\circ [|x| > 0; \; z(w) \geqslant 0]$ we define the controls

$$u^\circ(w^{(2)}, v) = 0, \quad v^\circ(w) = 0$$
$$w^{(2)}, w \in D_1^\circ = D^\circ \cap \{[\zeta = \xi^2 - y_\beta^2 < 0] \cup \tag{3.1}$$
$$\cup [\xi \geqslant 0, \quad y_\alpha - \sqrt{\zeta} > 0]\}$$
$$u^\circ(w^{(2)}, v) = -\sqrt{\zeta^{(2)}}\,\delta j_\alpha - |y_\beta^{(2)}|\,\delta j_\beta$$
$$v^\circ(w) \quad \text{is antiparallel to} \quad u^\circ(w, v) \tag{3.2}$$
$$w^{(2)}, w \in D_2^\circ = D^\circ \cap [\zeta \geqslant 0, y_\alpha - \sqrt{\zeta} \leqslant 0; \xi^2 + y_\beta^2 > 0]$$
$$u^\circ(w^{(2)}, v) = -|v_\alpha| j_\alpha - v_\beta j_\beta - v_\gamma j_\gamma$$
$$v^\circ(w) \quad \text{is arbitrary with the condition } v_\alpha \geqslant 0; \; [\nu_{2\alpha} \geqslant 0] \tag{3.3}$$
$$w \in D_3^\circ = D^\circ \cap [y_\alpha \leqslant 0; \quad \xi = |y_\beta| = 0]$$

The controls $v^\circ(w)$ in formulas (3.2), (3.3) are defined with a sufficient degree of arbitrariness. The second formula in (3.2) permits any finite or impulse control $v^\circ(w)$, antiparallel to vector $u^\circ(w, v)$, which does not depend on v in the case given. The second formula in (3.3) permits any finite or impulse control $v^\circ(w)$ with a nonnegative projection v_α $[\nu_{2\alpha}]$. The first formula in (3.3) in fact coincides with the first formula in (3.2) for a second player's impulse control realizing the inclusion $w^{(2)} \in D_2^\circ$.

Theorem 3.1. The pair of controls $u^\circ(w^{(2)}, v)$, $v^\circ(w)$ corresponds to Problem 1 and realizes in region D_1° the time

$$T[u^\circ, v^\circ] = T^\circ(w) = t_1(w) + \pi/2; \; w \in D_1^\circ$$

where $t_1(w)$ is the smallest positive root of the equation

$$\eta(w, t) = (x^2 - \xi^2)\sin^2 t - 2|x| y_\alpha \sin t \cos t + (y_\alpha^2 + y_\beta^2 - \xi^2)\cos^2 t = 0$$

while in the region $D_2^\circ \cup D_3^\circ$ the time

$$T[u^\circ, v^\circ] = T^\circ(w) = \pi/2 + \operatorname{arctg}(p_1(w)/|x|)$$
$$p_1(w) = y_\alpha - \sqrt{\xi^2 - y_\beta^2}$$

The time $T[u^\circ, v^\circ]$ corresponds to the estimate $T[u^\circ, v^\circ] \leqslant T[u^\circ, v]$ for any pair $u^\circ(w^{(2)}, v)$, $v(w)$ and to the estimate $T[u, v^\circ] \geqslant T[u^\circ, v^\circ]$ for any pair $u(w^{(2)}, v)$, $v^\circ(w)$, retaining the trajectory in region D° (*).

The proof of Theorem 3.1 requires the successive proofs of a number of the following Assertions.

3.1.1 (**). If for $w \in D_1^\circ$ both players use finite controls u, v, then the estimates

$$T^{\circ\cdot}(w, u^\circ = 0, v \neq 0) \leqslant T^{\circ\cdot}(w, u^\circ = 0, v^\circ = 0) = -1 \leqslant T^{\circ\cdot}(w, u \neq 0, v^\circ = 0) \quad (3.4)$$

are valid.

Proof. Setting, for the sake of brevity,

$$\sin(t_1(w)) = s, \quad \cos(t_1(w)) = c, \quad 2/(\partial\eta(w, t = t_1)/\partial t) = \psi$$

for the derivative $T^{\circ\cdot}$ we obtain the extression

$$T^{\circ\cdot} = -1 - \psi[\xi(-|u| + |v|) - (-y_\alpha c^2 + |x|cs)(u_\alpha + v_\alpha) + |y_\beta| c^2 (u_\beta + v_\beta)] \quad (3.5)$$

Since $t_1(w)$ is the smallest positive root of the equation $\eta(w, t) = 0$ and since the condition $\eta(w, 0) > 0$ is fulfilled, the estimate $\partial\eta(w, t = t_1)/\partial t \leqslant 0$ is necessarily fulfilled. Let us assume that the latter qunatity is negative. The factor in the brackets in formula (3.5) is nonpositive for $|u| \neq 0$, $|v| = 0$ and is nonnegative for $|u| = 0$, $|v| \neq 0$. To prove the last assertion it is sufficient to consider the expression

$$\xi^2 - (-y_\alpha c^2 + |x|cs)^2 - y_\beta^2 c^4 = \xi^2 s^2 > 0 \quad (3.6)$$

The second part of relation (3.6) is obtained from the first after taking into account the equality $\eta(w, t_1) = 0$. Thus, the estimate (3.6) proves estimates (3.4) for $\partial\eta(w, t = t_1)/\partial t < 0$.

The case $\partial\eta(w, t = t_1)/\partial t = 0$ requires additional analysis which briefly consists of the following. We can show that in this case the equality $z(w) = 0$ is fulfilled. The condition $w \in D_1^\circ$ implies as a consequence the estimate $z(w, u \neq 0, v = 0) < 0$, because of which the motion passes into region D_0 when $u \neq 0$. Any pair $u^\circ = 0$, $v \neq 0$ leads to a growth of the function $z(w)$, so that for any small finite segment of the trajectory corresponding to the pair $u^\circ = 0$, $v(w) \neq 0$ we can establish the estimate $\Delta T^\circ < \Delta t$ because for any $0 < t < \Delta t$ the derivative $T^{\circ\cdot}(w, 0, v)$ exists and satisfies the relation $T^{\circ\cdot}(w, 0, v) \to -\infty$ as $t \to +0$. The proof of assertion 3.1.1 is completed.

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*) Since the control $v^\circ(w)$ is defined only in D°, the pair $u(w, v)$, $v^\circ(w)$, retaining the trajectory in D° is called the pair on which the inclusion $w^{(2,1)}(t, \{u(w, v), v^\circ(w)\}$, $w(0) \in D^\circ) \in D^\circ$, is valid up to the instant of hitting onto M, while the velocity of the representative point on the boundary $z_1(w) = 0$ is directed either toward the interior of region D or along the boundary. In this sense Theorem 3.1 makes no claim of a complete correspondence with Problem 1.

**) Assertions 3.1.1, ..., 3.1.4 are concerned with finite pairs u, v°.

3.1.2. The estimate

$$T^{\circ\cdot}(w, u, v^\circ) > -1 \tag{3.7}$$

is valid for the region D_2° with the exception of the boundaries

$$D_{2,1}^\circ = [\zeta = 0;\ y_\alpha \leqslant 0] \cap D_2^\circ;\ D_{2,2}^\circ = [\zeta > 0,\ y_\alpha - \sqrt{\zeta} = 0] \cap D_2^\circ$$

To prove Assertion 3.1.2 we note that the estimate $\zeta > 0$ and the estimate $p_1(w) < 0$ are valid in the indicated part of region D_2°, and we write down the derivative $T^{\circ\cdot}$

$$T^{\circ\cdot} = -1 + (p_1^2 + x^2)^{-1}[-p_1\xi^2 / \sqrt{\zeta} + R_1(w, u, v^\circ)] \tag{3.8}$$

$$R_1^\circ(w, u, v^\circ) = |x|(+u_\alpha + (\xi|u| + |y_\beta|u_\beta) / \sqrt{\zeta}) \tag{3.9}$$

The obvious estimate $R_1(w, u, v^\circ) \geqslant 0$, the estimate $p_1(w) < 0$, and the formula (3.8) prove estimate (3.7) and Assertion 3.1.2.

3.1.3. On the boundary $\overset{\circ}{D}_{2,1}$ any pair $u \neq u^\circ(w, v)$, $v^\circ(w)$ either transfers the trajectory into region D_1° with a positive jump $\Delta T^\circ > 0$ or transfers the trajectory into region D_0.

Proof. Let $\zeta = 0$, $y_\alpha < 0$. Under these conditions

$$\xi - |y_\beta| = 0, \quad \xi^\cdot - |y_\beta|^\cdot = -|u| + |v^\circ| - (u_\beta + v_\beta^\circ) + y_\alpha|y_\beta| / |x| < 0$$
$$\text{for}\ \ y_\alpha < 0,\ \ |y_\beta| > 0$$

This means that there exists a unique control $u^\circ(w, v) = -|y_\beta|\delta/_\beta$ preserving the equality $\xi(t > 0) - |y_\beta(t > 0)| = 0$, while any other one transfers the position, for $y_\alpha < 0$, into region D_1° or into region D_0. The remaining cases $y_\alpha = 0$ or $y_\beta = 0$ are investigated analogously.

3.1.4. The equality $p_1(w) = 0$ is fulfilled on the boundary $\overset{\circ}{D}_{2,2}$ and, therefore, on the basis of formulas (3.8), (3.9) we can assert that the equality $T^{\circ\cdot}(w, u, v^\circ) = -1$ is valid for any control u parallel to the optimal jump $u^\circ(w, v)$ and for $v = v^\circ$. However, it is obvious that for $u \neq u^\circ(w, v)$ the equality $T^{\circ\cdot}(w, u, v^\circ) = -1$ is violated at an infinitely close adjacent position and turns into the inequality

$$T^{\circ\cdot}(w + \Delta w,\ u,\ v^\circ) > -1$$

3.1.5. Suppose that for $w \in D_2^\circ$ the second player realizes a certain optimal jump $v^\circ(w)$ antiparallel to the vector $u^\circ(w, v)$, while the first player realizes the jump $mu^\circ(w^{(2)}, v)$ with $0 \leqslant m < 1$. Simple calculations show the validity of the equalities

$$p_1(w) = p_1(w^{(2)}) = p_1(w^{(2,1)})$$
$$\xi / \sqrt{\zeta} = \xi^2 \sqrt{\zeta^{(2)}} = \xi^{(2,1)} / \sqrt{\zeta^{(2,1)}}$$

If after the indicated jumps both players realize the controls u_1, v_1°, parallel to the optimal jumps, then formula (3.8) shows that the derivative $T^{\circ\cdot}(w^{(2,1)}, u, v)$ proves to be closer to -1 the lesser is the quantity $\xi^{(2,1)}$. This allows us to obtain the equality

$$\lim_{m \to 1} T^{\circ\cdot}(w^{(2,1)}(w, mu^\circ, v^\circ), u_1, v_1^\circ) = -1 \tag{3.10}$$

3.1.6. Suppose that at $t = 0$ both protagonists have realized optimal jumps (then $w^{(2,1)} \in D_3^\circ$, and finite controls are realized for $t > 0$). The derivative T has the form $T^{\circ\cdot} = -1 + |x|(x^2 + p_1^2)^{-1}\{(u_\alpha + v_\alpha - [(|u| - |v|)^2 - (u_\beta + v_\beta)^2 - (u_\gamma + v_\gamma)^2]^{1/2}\}$

From the latter formula we see that any pair u°, $v \neq v^\circ$ decreases the derivative T°

from -1, while any pair $u \neq u^\circ$, v° either increases the derivative $T^{\circ\cdot}$ from -1 or transfers the position into region D_0.

Assertions 3.1.1 – 3.1.6 settle the proof of Theorem 3.1 for those cases when the players realize jumps parallel to the optimal ones or use finite controls.

3.1.7. Any impulse control $v = \nu_2\delta \neq v^\circ(w)$ strictly decreases the function $T^\circ(w)$, i.e. $T^\circ(w^{(2)}) < T^\circ(w)$.

To prove Assertion 3.1.7 we should inspect a number of cases.

3.1.7.1. Let $w \in D_1^\circ$, $w^{(2)} \in D_1^\circ$. We consider the function

$$\eta(w^{(2)}, t_1(w)) = -|\nu_2|^2 s^2 - 2[\xi|\nu_2| + |x|\nu_{2\alpha} cs - (y_\alpha \nu_{2,\alpha} + |y_\beta|\nu_{2,\beta})c^2] < 0$$

The latter estimate follows from the estimates $|\nu_2| > 0$; $s^2 = \sin^2(t_1(w)) > 0$ and from estimate (3.6). From the estimates $\eta(w^{(2)}, t_1(w)) < 0$; $\eta(w^{(2)}, 0) > 0$ there follows the estimate $\Delta T^\circ = t_1(w^{(2)}) - t_1(w) < 0$ which completes the proof of Assertion 3.1.7 in the case 3.1.7.1.

3.1.7.2. Let $w \in D_2^\circ \cup D_3^\circ$, $w^{(2)} \in D_2^\circ \cup D_3^\circ$. We consider the difference

$$\Delta p_1 = p_1(w^{(2)}) - p_1(w) = \nu_{2,\alpha} - \sqrt{\zeta + \nu_{2,\alpha}^2 + 2(\xi|\nu_2| - |y_\beta|\nu_{2,\beta})} + \sqrt{\zeta}$$

The difference Δp_1 can become a nonnegative number only if the estimates

$$\sqrt{\zeta} + \nu_{2,\alpha} \geqslant 0. \quad -\xi|\nu_2| + \nu_{2,\alpha}\sqrt{\zeta} + \nu_{2,\beta}|y_\beta| \geqslant 0$$

are fulfilled simultaneously. Both these estimates are compatible only for $v = v^\circ(w)$. The estimate $\Delta p_1 < 0$ and the estimate $\Delta T^\circ < 0$ are realized for $v = \nu_2\delta \neq v^\circ(w)$; this completes the examination of case 3.1.7.2.

3.1.7.3. Let $w \in D_1^\circ$, $w^{(2)} \in D_2^\circ \cup D_3^\circ$. The impulse $v = \nu_2\delta$ can lead to a crossing of the boundary $\overset{\circ}{D}_{2,1}[\zeta = 0;\ y_\alpha \leqslant 0]$. With this crossing the function $T^\circ(w)$ undergoes a negative jump. Under a crossing of the boundary $\overset{\circ}{D}_{2,2}[\zeta > 0,\ y_\alpha - \sqrt{\xi} = 0]$ the function $T^\circ(w)$ is continuous. These two facts, together with the results in 3.1.7.1, 3.1.7.2, allows us to prove Assertion 3.1.7 in the case 3.1.7.3.

3.1.8. There is no impulse control $v = \nu_2\delta$ which can effect the following crossings

$$[w \in D_2^\circ \cup D_3^\circ;\ w^{(2)} \in D_1^\circ],$$
$$[w \in D^\circ;\ w^{(2)} \in D_0]$$

The proof is carried out on the basis of the estimate $\Delta p_1 \leqslant 0$ and of Lemma 2.1.

3.1.9. Any impulse control $u = \mu_1\delta \neq mu^\circ(w, v)$ $(0 \leqslant m \leqslant 1)$ either strictly increases the function $T^\circ(w)$ or transfers the position into region D_0; $(T^\circ(w^{(1)}) > T^\circ(w))$.

The proof is carried out in analogy with the proof of Assertion 3.1.7, by examining in succession the cases

3.1.9.1 $w \in D_1^\circ$, $w^{(1)} \in D_1^\circ$
3.1.9.2 $w \in D_2^\circ \cup D_3^\circ$; $w^{(1)} \in D_2^\circ \cup D_3^\circ$
3.1.9.3 $w \in D_2^\circ \cup D_3^\circ$, $w^{(1)} \in D_1^\circ$

3.1.10. There is no impulse control $u = \mu_1\delta$ which can effect the crossing

$$[w \in D_1^\circ;\ w^{(1)} \in D_2^\circ \cup D_3^\circ].$$

The proof is based on the estimate $\eta(w^{(1)}, 0) > 0$. The agregate of Assertions 3.1.1 - 3.1.10 proves Theorem 3.1.

4. Let us give a short geometrical interpretation of the optimal motion. Let the representative point whose position is determined by the vector x describe, in a control-free motion, an ellipse (Fig. 1). The arcs ac and a_1c_1 of the ellipse belong to region $D_1°$. On these arcs the first player either (on the segment ab) cannot cancel the lateral component $|y_\beta| > \xi$ or (on the segment bc) passes to the value $y_\alpha^{(1)}(w) > 0$ $(p_1(w) > 0)$. This is valid, of course, for controls u preserving the inclusion $w^{(1)} \in D°$. The arguments set forth force both players to follow along the arc (ac) with controls $u° = v° = 0$ up to the point c at which the equality $\xi = |y|$ is first realized. At the point c the first player applies the impulse $u°(w^{(2)}, v) = -y\delta$ and the subsequent motion takes place along the straight line $c\,0$. The equality $\xi = |y(t_1)|$ is equivalent to the equation $\eta(w, t_1) = 0$ for $w \in D_1°$, and the total time for hitting into the point O $(|x| = 0)$ turns out to be $t_1(w) + \pi/2 = T°(w)$.

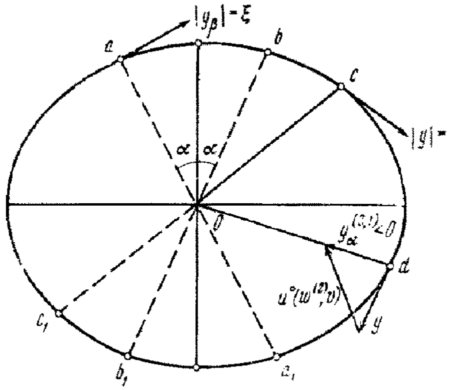

Fig. 1

However, if the position $w \in D_2°$, which corresponds to locations on the arcs $[c, a_1]$, $[c_1, a]$, then at any point d of the arc $[c, a_1]$ the optimal control $u°(w^{(2)}, v)$ is realized in the form of an impulse (n, m), while the radial velocity $y_\alpha^{(2,1)}(w^{(2)})$ obtained proves to be nonpositive. The subsequent motion takes place along the straight line $(d, 0)$ and the hitting into the origin 0 is realized after a time $T(w) = \pi/2 + \operatorname{arc\,tg}(p_1(w)/|x|)$. If at any point a the first player does not cancel the velocity y_β with the impulse $u° = -y_\beta \delta j_\beta$, but applies some finite control $u(w)$, then the position passes into region $D_1°$ and the time $T(w)$ increases by a jump.

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